Lecture Notes: Probability Set 3

Ashoka University

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Tower Property of Conditional Expectation

Theorem (Tower Property)

Let $\mathcal{H}, \mathcal{G}, \mathcal{F}$ be σ-algebras such that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$. Let X be an rv on $(\Omega, \mathcal{F}, P)$. Then $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$.

- **Proof:** We want to find r.v Y s.t. it is $\mathcal{H}$-m'sble, and

$$\int_H Y dP = \int_H E[X|\mathcal{G}] dP \ \ \forall H \in \mathcal{H}$$

- But $\int_H E[X|\mathcal{G}] dP = \int_H X dP \ \ \forall H \in \mathcal{H}$.
- And $\int_H E[X|\mathcal{H}] dP = \int_H X dP, \quad \forall H \in \mathcal{H}$
- Since Y is unique, it follows that $Y = E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$ a.s

Tower Property: Example

Example

Let $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be s.t. $\mathcal{F} = \sigma(X, Y, Z)$, $\mathcal{G} = \sigma(Y, Z)$, $\mathcal{H} = \sigma(Z)$.
Then $E[X|\mathcal{G}]$ becomes $E[X|Y, Z]$.
Assuming X, Y, Z take discrete values,

$$E[X|Y = y_j, Z = z_l] = \sum_k x_k P(X = x_k|Y = y_j, Z = z_l)$$

Now, we compute $E[E[X|Y, Z]|Z = z_l]$:

$$= \sum_j \left(\sum_k \frac{x_k P(X = x_k, Y = y_j, Z = z_l)}{P(Y = y_j, Z = z_l)} \right) P(Y = y_j|Z = z_l)$$

$$\therefore = \sum_j \sum_k \frac{x_k P(X = x_k, Y = y_j, Z = z_l)}{P(Z = z_l)}$$

Continuing from the previous step

Example

$$E[E[X|Y,Z]|Z=z_l] = \sum_j \sum_k \frac{x_k P(X=x_k, Y=y_j, Z=z_l)}{P(Z=z_l)}$$

Changing the order of summations we get:

$$= \sum_k \frac{x_k \sum_j P(X=x_k, Y=y_j, Z=z_l)}{P(Z=z_l)}$$

$$= \sum_k \frac{x_k P(X=x_k, Z=z_l)}{P(Z=z_l)} = E[X|Z=z_l]$$

Property: Taking out what is known

Theorem

Let Z be an rv, $\mathcal{G}$-measurable. Then $E[XZ|\mathcal{G}] = ZE[X|\mathcal{G}]$.

- **Proof:** Let $Z = \mathbb{1}_H, \quad H \in \mathcal{G}. \ \therefore E[XZ|\mathcal{G}] = E[X \cdot \mathbb{1}H|\mathcal{G}]$.
- Let Y be s.t. $\int_G YdP = \int_G XZdP, \quad G \in \mathcal{G}$.

$$\int_G YdP = \int_G X\mathbb{1}_H dP = \int_{G\cap H} XdP \quad (G \cap H \in \mathcal{G})$$

$$= \int_{G\cap H} E[X|\mathcal{G}]dP = \int_G E[X|\mathcal{G}] \cdot \mathbb{1}_H dP$$

- Thus we have shown $Y = ZE[X|\mathcal{G}]$, when Z is a indicator fn.

Proof for Taking out what is known (Cont.)

- Suppose $X \geq 0$. Z_n a simple non-negative function that increases to $Z \geq 0$.
- We have $E[XZ_n|\mathcal{G}] = Z_n E[X|\mathcal{G}]$.
- By MCT $E[XZ|\mathcal{G}] = ZE[X|\mathcal{G}]$
- **Last step:** X and Z need not be non-negative. We can always decompose X and Z as: $X = X^+ - X^-$, $Z = Z^+ - Z^-$ and then replicate the proof for non-negative X and Z on the component pairs.

Conditional expectation when X is independent of $\mathcal{G}$

- Suppose $X = I_F$ and F is independent of all sets in $\mathcal{G}$. Then,

$$\int_G I_F dP = P(G \cap F) = P(G)P(F) = \int_G P(F) dP$$

Thus, $E(X|\mathcal{G}) = EX$ when X is an indicator function. The proof completes with standard machinery.

Martingale

Definition

A sequence of r.v's $M_n : n \geq 0$ is called a martingale if:

a) $E|M_n| < \infty \quad \forall n \geq 0$.

b) $E[M_{n+1}|M_1, M_2, \ldots, M_n] = M_n, \quad \forall n \geq 0$.

Definition

Let $\mathcal{F}_n : n \geq 1$ be a collection of σ-algebras such that $\mathcal{F}_n \subset \mathcal{F}_{n+1} \quad \forall n \geq 1$. Let X_n, for each n, be a rv that is $\mathcal{F}_n$-measurable. Then $(\mathcal{F}_n : n \geq 1)$ is said to be the filtration and $(X_n : n \geq 1)$ is said to be a process adapted to the filteration.

Intuitively, $\mathcal{F}_n$ can be thought of collection of info available at time n, for example winnings in a betting game for n rounds.

Martingale definition updated

Definition

A sequence of r.v's $M_n : n \geq 0$ adapted to $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$ (where $\mathcal{F}_0 \subset \mathcal{F}1 \subset ... \subset \mathcal{F}$) is called a martingale if:

a) $E|M_n| < \infty \quad \forall n \geq 0.$

b) $E[M_{n+1}|\mathcal{F}_n] = M_n, \quad \forall n \geq 0.$

Example: Mean Zero Random Walk

- Let $S_n = \sum_{i=1}^{n} X_i$, where $E[X_i] = 0 \quad \forall i$, and X_i's are independent.
  a) Check integrability: $E|S_n| = E|\sum_1^n X_i| \leq \sum_{i=1}^{n} E|X_i| < \infty \quad \checkmark$
  b) Check martingale property:

$$E[S_{n+1}|(X_1, \ldots, X_n)] = E[S_n|(X_1, \ldots, X_n)] + E[X_{n+1}|\mathcal{F}_n] \quad = S_n$$

- $\therefore S_n, n \geq 1$ is a martingale.(Here, $\sigma(X_1, \ldots, X_n) \subset \mathcal{F}_n$ for each n).

Example 2: Mean Zero Random Walk

- Let $S_n = \sum_{i=1}^{n} X_i$, where $E[X_i] = 0 \quad \forall i$, and X_i's are independent. $Var(X_i) = \sigma^2 < \infty$.
- Let $M_n = S_n^2 - n\sigma^2$.
  - a) Check martingale property:

$$\begin{aligned} E[M_{n+1}|(X_1, \ldots, X_n)] &= E[(S_n + X_{n+1})^2 - (n+1)\sigma^2|(X_1, \ldots, X_n)] \\ &= S_n^2 + EX_{n+1}^2 - (n+1)\sigma^2 = M_n \end{aligned}$$

- $\therefore M_n, n \geq 1$ is a martingale.

Example 2: Product Martingale

Example

Let $M_n = \prod_{i=1}^{n} X_i$, where $E[X_i] = 1, X_i \geq 0 \quad \forall i$, and X_i's are independent.

a) $E|M_n| = E|X_1 X_2 ... X_n| = E|X_1| \cdot E|X_2| \cdots E|X_n| < \infty \quad \checkmark$

b)

$$\begin{aligned} E[M_{n+1}|\mathcal{F}_n] &= E[M_n \cdot X_{n+1}|\mathcal{F}_n] \\ &= M_n E[X_{n+1}|\mathcal{F}n] \\ &= M_n E[X_{n+1}] \quad (\because X_{n+1} \& \mathcal{F}_n \text{ are indept.}) \\ &= M_n. \quad \checkmark \end{aligned}$$

Thus M_n is a martingale.

Martingale Convergence Theorem

Theorem (Martingale Convergence Theorem)

If M_n is a martingale such that $\sup_n E|M_n| < \infty$, then $\exists M_\infty$ s.t. $M_n \to M_\infty$ a.s.

A powerful theorem to prove non-trivial results

Example 3: Doob's Martingale

- Let X be a rv on $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$ such that $E|X| < \infty$.
- Then $Y_n = E[X|\mathcal{F}_n]$ is a martingale.
- $E|E[X|\mathcal{F}_n]| \leq E[E[|X||\mathcal{F}n]] = E|X| < \infty$ (Conditional Jensen's) $\forall n$ $\Rightarrow \sup_n E|Y_n| < \infty$ — (A)
- $E[Y_{n+1}|\mathcal{F}n] = E[E[X|\mathcal{F}_{n+1}]|\mathcal{F}_n] = E[X|\mathcal{F}n]$ (By tower property) $= Y_n$ — (B)
- (A) (B) imply $\exists Y_\infty$ s.t. $Y_n = E[X|\mathcal{F}n] \to Y_\infty$ as $n \to \infty$ (By Martingale Convergence Theorem).

Example 4: Martingale Transform

Example

Given a martingale M_n, one can also construct other martingales from that.

- C_n is a predictable process if C_n is $\mathcal{F}_{n-1}$-measurable $\forall n$.
- Let M_n be a martingale defined over $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$. Fix $M_0 = c$.
- Let $Y_n = \sum_{k=1}^{n} C_k(M_k - M_{k-1})$.
- Assume $|C_n| \leq K \quad \forall n$.

Then Y_n will be a martingale.

Example 4: Proof (Integrability)

Proof.

First, we check for integrability:

$$
\begin{aligned}
E|Y_n| &= E\left|\sum_{k=1}^{n} C_k(M_k - M_{k-1})\right| \\
&\leq \sum_{k=1}^{n} E|C_k(M_k - M_{k-1})| \\
&\leq \sum_{k=1}^{n} KE|M_k - M_{k-1}| \\
&\leq \sum_{k=1}^{n} K(E|M_k| + E|M_{k-1}|) \\
&< \infty \quad (\because E|M_k| < \infty \quad \forall k) \quad \checkmark
\end{aligned}
$$

□

Example 4: Proof (Martingale Property)

Proof cont.

Next, we check the martingale property:

$$E[Y_n - Y_{n-1}|\mathcal{F}_{n-1}] = E[C_n(M_n - M_{n-1})|\mathcal{F}_{n-1}]$$

$$= C_n E[M_n - M_{n-1}|\mathcal{F}_{n-1}]$$

$$= C_n(M_{n-1} - M_{n-1}) = 0$$

$$\Rightarrow E[Y_n|\mathcal{F}_{n-1}] = Y_{n-1} \quad \checkmark$$

□

This process, known as a martingale transform, can be thought of as a discrete analog of a stochastic integral.

Martingale Transform: Intuition

- C_n can be thought of as the amount one has decided to bet in n^{th} round of a betting game which is determined by the winnings of last $n-1$ rounds, thus making C_n $\mathcal{F}_{n-1}$-measurable;

- $(M_n - M_{n-1})$ can be thought of the winning in n^{th} round.

Observe that

$$E[M_n|\mathcal{F}_{n-1}] = M_{n-1} \quad \forall n$$

$$\Rightarrow E[M_n] = E[M_{n-1}] \quad \forall n.$$

Stopping Time

Example

Let $S_n = X_1 + X_2 + ... + X_n$ where $X_i = \pm 1$ w.p. $1/2, \forall i$ denotes a simple, symmetric random walk. Let $\tau = \inf\{n : S_n = +1\}$. τ is a rv that is also a stopping time.

It can be shown that $P(\tau < \infty) = 1$, but $E[\tau] = \infty$.

Definition

rv τ is a stopping time if $\{\omega : \tau(\omega) \leq n\}$ is $\mathcal{F}_n$-msble for each n. (Whether or not to stop the game at n depends on the information available till time n).

Stopped process

- Consider again the martingale

$$Y_n = \sum_{k=1}^{n} C_k(M_k - Mk - 1)$$

w.r.t. filtration $\{\mathcal{F}_n\}$.

- Set $C_n = I(\tau \geq n)$. Then, $Y_n = M_{n \wedge \tau}$ is a martingale w.r.t. filtration $\{\mathcal{F}_n\}$.

- Consider stopping time

$$\tau = \min\{n : S_n = A \text{ or } S_n = -B\}$$

where $\{S_n\}$ is again a simple symmetric random walk and A, B are positive integers.

Level crossing probability

- To find the probability $P(S_\tau = A)$.
- $\{S_{n\wedge\tau}\}$ is a martingale
- $\{S_{n\wedge\tau}\} \to S_\tau$ as $n \to \infty$ (since, $P(\tau < \infty) = 1$).
- By bounded convergence theorem, $ES_{n\wedge\tau} \to ES_\tau$ as $n \to \infty$.
- So $ES_\tau = 0$.
- $AP(S_\tau = A) - BP(S_\tau = -B) = 0$ so that

$$P(S_\tau = A) = \frac{B}{A+B}$$

Finding $E\tau$

- Consider the martingale $M_n = S_n^2 - n$
- Again $\{M_{n\wedge\tau}\}$ is a martingale
- $M_{n\wedge\tau} \to M_\tau$ as $n \to \infty$ (since, $P(\tau < \infty) = 1$).
- Further, $M_{n\wedge\tau} \leq A^2 + B^2 + \tau$ and $E[\tau] < \infty$. By dominated convergence theorem, $EM_\tau = EM_1 = 0$

$$A^2 P(S_\tau = A) + B^2 P(S_\tau = B) - E\tau = 0.$$

Or

$$E\tau = AB.$$

$E\tau < \infty$

- To see that $E\tau < \infty$, recall that

$$P(\tau \geq n(A+B)) \leq a^n$$

where

$$a = 1 - \left(\frac{1}{2}\right)^{A+B}.$$

Now for nonnegative integer τ,

$$E\tau = \sum_{n=1}^{\infty} P(\tau \geq n).$$

This implies that

$$E\tau < (A+B) + (A+B)\sum_{n=1}^{\infty} a^n < \infty.$$

Martingale Stopping Time Theorem

1. Let M_n be a martingale on $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$, τ be the stopping time. $M_0 = c$.

2. Then $E[M_\tau] = E[M_1]$ if:

    1. τ is bounded a.s. ($\exists K < \infty$ s.t. $P(\tau < K) = 1$)

    2. $|M_n| \leq K \quad \forall n$, and $P(\tau < \infty) = 1$. (by DCT, $M_n \uparrow M_\tau$ as $n \uparrow \infty$, $E[M_n] \uparrow E[M_\tau]$)

    3. $|M_n - M_{n-1}| \leq K \quad \forall n \geq 1$, and $E[\tau] < \infty$.

3. (note that $M_n = \sum_{k=1}^{n-1}(M_k - M_{k-1}$, so $M_\tau \leq c + K\tau$)

Martingale example

- Alphabet symbols are picked independently, each of the 26 equally likely. How many draws on average until one sees ABRACADABRA.
- Consider a gamble where a new party bets Rs.1 at each time. If A comes at the bet, the winning party wins 25 rupees and bets all of Rs 26 on B. If now B comes, it bets $26 + 25 \times 26$ on R and so on. Whenever this party loses, its overall loss equals Rs. 1.
- Total winning of all parties at any time n is a martingale, and expected total till time n is zero.
- Suppose ABRACADABRA comes after time T. The total winnings of all parties equals
$$26^{11} + 26^{11} + 26 - T$$
- Thus, $ET = 26^{11} + 26^{11} + 26$.

Submartingales

- A sequence of r.v's $M_n : n \geq 0$ adapted to $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$ (where $\mathcal{F}_0 \subset \mathcal{F}1 \subset ... \subset \mathcal{F}$) is called a sub-martingale (super-martingale) if:

  a) $E|M_n| < \infty \quad \forall n \geq 0.$

  b) $E[Mn + 1|\mathcal{F}_n] \geq (\leq) M_n, \quad \forall n \geq 0.$

Submartingale Inequality

- Let $(M_n : n \geq 0)$ be a non-negative submgle. $M_n^* = \sup_{0 \leq m \leq n} M_m$. Then, for $a > 0$,
$$P(M_n^* \geq a) \leq \frac{EM_n}{a}.$$

- **Pf.** Let $A = \{M_n^* \geq a\}$. Then $A = \cup_{0 \leq m \leq n} A_m$,

- where $A_0 = \{M_0^* \geq a\}$, $A_m = \{M_{m-1}^* < a, M_m \geq a\}$ for each m.

- Now
$$P(A_m) \leq \int_{A_m} \frac{M_m}{a} dP \leq \int_{A_m} \frac{M_n}{a} dP$$

- So,
$$P(M_n^* \geq a) = \sum_{m=0}^{n} P(A_m) \leq \int_A \frac{M_n}{a} dP \quad \square$$