Stochastic Calculus Mathematical Finance

Brownian Motion

Fall 2025

Brownian Motion: Definition

Definition

A stochastic process $\{B(t) : 0 \leq t \leq T\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is defined as a Brownian Motion if it satisfies the following:

1. $B_0 = 0$
2. $B(t_4) - B(t_3)$ is independent of $B(t_2) - B(t_1)$ for all $0 \leq t_1 < t_2 < t_3 < t_4 \leq T$
3. $B(t+s) - B(s) \sim N(0, t)$ for all $s, t \geq 0$
4. $B(t, \omega)$ is a continuous function of t for every ω

Construction of BM: Simple Random Walk

- Let $X_1, X_2, \dots, X_n$ be i.i.d, $X_i = \pm 1$ w.p. $1/2$.
- Define $S_n = \sum_{i=1}^{n} X_i$ (Which is a simple random walk).
- Easy to see that $\{S_n : n \geq 1\}$ is a martingale and is a Markov process.

Markov Property in discrete settings

For $\{S_n : n \geq 1\}$ to be Markov, need to show that

$$\mathbb{P}[S_{n+1} = s_{n+1} | S_0 = s_0, S_1 = s_1, \ldots, S_n = s_n] = \mathbb{P}[S_{n+1} = s_{n+1} | S_n = s_n]$$

We have

$$\mathbb{P}[S_{n+1} = s_{n+1} | S_0 = s_0, S_1 = s_1, \ldots, S_n = s_n]$$

$$= \mathbb{P}[S_n + X_{n+1} = s_{n+1} | S_0 = s_0, S_1 = s_1, \ldots, S_n = s_n]$$

$$= \mathbb{P}[X_{n+1} = s_{n+1} - s_n | S_0 = s_0, \ldots, S_n = s_n]$$

$$= \mathbb{P}[X_{n+1} = s_{n+1} - s_n] \quad \text{(since } X_i\text{'s are independent)}$$

$$= \mathbb{P}[S_{n+1} = s_{n+1} | S_n = s_n]$$

Scaled Random Walk: Definition

- Define another process $\{B_n(t) : 0 \leq t \leq T\}$ such that:

$$B_n(t) = \frac{S_{nt}}{\sqrt{n}}$$

  when t is a multiple of n^{-1}, and a linear interpolation at other points.

- Time is scaled down by n and space by $\sqrt{n}$.
  $\{B_n(t) : 0 \leq t \leq T\}$ is constructed

- We argue that $\{B_n(t) : 0 \leq t \leq T\} \to \{B(t) : 0 \leq t \leq T\}$.
  (This is known as the weak convergence of stochastic processes).

Scaled Random Walk: Properties (1/2)

1. $$B_n(0) = \frac{S_0}{\sqrt{n}} = 0 \qquad \text{(By defn. of r.w.)} \tag{1}$$

2. Let $0 < t_1 < t_2 < t_3 < t_4 < T$. Then, roughly speaking

$$B_n(t_4) - B_n(t_3) = \frac{S_{nt_4} - S_{nt_3}}{\sqrt{n}} = \frac{\sum_{i=nt_3+1}^{nt_4} X_i}{\sqrt{n}}$$

$$B_n(t_2) - B_n(t_1) = \frac{S_{nt_2} - S_{nt_1}}{\sqrt{n}} = \frac{\sum_{i=nt_1+1}^{nt_2} X_i}{\sqrt{n}}$$

Since X_i's are independent, asymptotically, for large n, $B_n(t_4) - B_n(t_3) \perp\!\!\!\perp B_n(t_2) - B_n(t_1)$.

Scaled Random Walk: Properties (2/2)

3. $B_n(t+s) - B_n(s) = \frac{S_{n(t+s)} - S_{ns}}{\sqrt{n}} = \frac{\sum_{i=ns+1}^{rn(t+s)} X_i}{\sqrt{n}}$

   $= \text{ in dist } \sqrt{t}\left(\frac{\sum_1^{nt} X_i}{\sqrt{nt}}\right) \xrightarrow{d} N(0,t)$ (By Central Limit Theorem)

4. It can be verified that the limiting process $B(t,\omega)$ is continuous in t for each ω. In fact, it can be shown that it is continuous everywhere, but differentiable nowhere.

Brownian Motion is a Martingale

For $t, s \geq 0$:

$$\mathbb{E}[B_{t+s}|\mathcal{F}_t] = \mathbb{E}[B_{t+s} - B_t + B_t|\mathcal{F}_t]$$

By property (2) of BM, $B_{t+s} - B_t \perp\!\!\!\perp \mathcal{F}_t$.

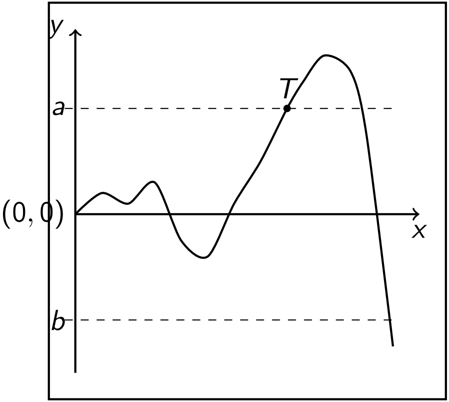
$$\implies \mathbb{E}[B_{t+s} - B_t|\mathcal{F}_t] = \mathbb{E}[B_{t+s} - B_t]$$

$$\implies \mathbb{E}[B_{t+s}|\mathcal{F}_t] = \mathbb{E}[B_{t+s} - B_t] + B_t = 0 + B_t = B_t$$

Also, it can be verified that $\mathbb{E}[|B_t|] < \infty$ [since its mean of $|Z|$, where $Z \sim N(0, 1)$].

Stopping Time: Definition

Let $\tau = \inf\{t : B_t = a \text{ or } B_t = -b\}$, for $a, b > 0$.



Let's define the stopped process $Y_t = B_{t \wedge \tau}$.

Application 1: Exit Probability

Calculating the Probability

Since $B_{t\wedge\tau}$ is a martingale, $\mathbb{E}[B_{t\wedge\tau}] = \mathbb{E}[B_0] = 0$ for all t.

$$\implies \mathbb{E}[B_\tau] = 0$$

Now, we expand the expectation:

$$\mathbb{E}[B_\tau] = a\mathbb{P}[B_\tau = a] + (-b)\mathbb{P}[B_\tau = -b] = 0$$

- We know $\mathbb{P}[B_\tau = a] + \mathbb{P}[B_\tau = -b] = 1$.
- Substituting gives: $a\mathbb{P}[B_\tau = a] - b(1 - \mathbb{P}[B_\tau = a]) = 0$.
- Solving for the probability:

$$(a+b)\mathbb{P}[B_\tau = a] = b \implies \mathbb{P}[B_\tau = a] = \frac{b}{a+b}$$

Another Martingale: $M_t = B_t^2 - t$

Proof

$$\begin{aligned}
\mathbb{E}[M_{t+s}|\mathcal{F}_t] &= \mathbb{E}[B_{t+s}^2 - (t+s)|\mathcal{F}_t] \\
&= \mathbb{E}[((B_{t+s} - B_t) + B_t)^2|\mathcal{F}_t] - (t+s) \\
&= \mathbb{E}[(B_{t+s} - B_t)^2] + B_t^2 + 2B_t\mathbb{E}[B_{t+s} - B_t] - (t+s) \\
&= s + B_t^2 + 0 - (t+s) = B_t^2 - t = M_t
\end{aligned}$$

Application 2: Wald's Identity

Expected Hitting Time

The process $\{B^2_{t\wedge\tau} - (t\wedge\tau)\}$ is a martingale. Applying martingale stopping theorem:

$$\mathbb{E}[B_\tau^2 - \tau] = 0 \implies \mathbb{E}[\tau] = \mathbb{E}[B_\tau^2]$$

We expand the expectation:

$$\begin{aligned}
\mathbb{E}[\tau] &= a^2\mathbb{P}[B_\tau = a] + (-b)^2\mathbb{P}[B_\tau = -b] \\
&= a^2\left(\frac{b}{a+b}\right) + b^2\left(1 - \frac{b}{a+b}\right) \\
&= a^2\frac{b}{a+b} + b^2\frac{a}{a+b} = \frac{ab(a+b)}{a+b} = ab
\end{aligned}$$

The Exponential Martingale

Definition and Proof

Define $M_t = e^{\theta B_t - \frac{\theta^2 t}{2}}$, for $\theta \in \mathbb{R}$.

$$\begin{aligned}
\mathbb{E}[M_{t+s}|\mathcal{F}_t] &= \mathbb{E}[e^{\theta B_{t+s} - \frac{\theta^2(t+s)}{2}}|\mathcal{F}_t] \\
&= e^{\theta B_t - \frac{\theta^2 t}{2}} e^{-\frac{\theta^2 s}{2}} \mathbb{E}[e^{\theta(B_{t+s}-B_t)}] \\
&= M_t \cdot e^{-\frac{\theta^2 s}{2}} \cdot e^{\frac{\theta^2 s}{2}} \quad (\text{MGF of } N(0,s)) \\
&= M_t
\end{aligned}$$

Therefore $\{M_t, t \geq 0\}$ is a martingale, and $\mathbb{E}[M_T] = \mathbb{E}[M_0] = 1$.

Change of Measure

Radon-Nikodym Derivative

Define a new measure $\tilde{\mathbb{P}}$ to be: $\tilde{\mathbb{P}}(A) = \mathbb{E}[M_T \cdot 1_A]$ where $M_T \geq 0$ and $\mathbb{E}[M_T] = 1$.

- In discrete set-up: $\sum_{\omega \in \Omega} M_T(\omega)\mathbb{P}(\omega) = 1$.
- Let $A = \{\omega\} \implies \tilde{\mathbb{P}}(\omega) = M_T(\omega)\mathbb{P}(\omega)$.
- This means $M_T(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{P(\omega)}$.
- More generally, $M_T = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega)$ a.s., is a Radon-Nikodym derivative.

As we will see, Girsanov's Theorem, uses appropriate M_T to relate Brownian motions under different measures.

The Gaussian example

- Consider rv $B_1, B_2, \ldots, B_T$ such that $B_1 = X_1 \sim N(0,1)$ and each increment $B_m - B_{m-1} = X_m \sim N(0,1)$ independent of $(B_1, \ldots, B_{m-1})$ under probability measure $\mathbb{P}$.

- Then, letting $x = (x_1, \ldots, x_T)$, $E_{\mathbb{P}}[H(B_1, \ldots, B_T)]$ equals

$$\int_{x \in \mathbb{R}^T} H\left(x_1, \ldots, \sum_{i=1}^{T} x_i\right) f(x_1) \ldots f(x_T) dx_1 \ldots dx_T$$

  where f is a pdf of $N(0,1)$.

- Let g be a pdf of $N(\mu, 1)$ and $\tilde{\mathbb{P}}$ be the probability measure under which the density f under $\mathbb{P}$ is replaced by g.

The Gaussian example

- We can re-express $E_P[H(B_1, \ldots, B_T)]$ as

$$\int_{x \in \mathbb{R}^T} H\left(x_1, \ldots, \sum_{i=1}^{T} x_i\right) \frac{f(x_1)\ldots f(x_T)}{g(x_1)\ldots g(x_T)} g(x_1)\ldots g(x_T) dx_1 \ldots dx_T$$

- Thus,

$$E_P[H(B_1, \ldots, B_T)] = E_{\tilde{\mathbb{P}}}[M_T H(B_1, \ldots, B_T)]$$

where

$$M_T = \frac{f(x_1)\ldots f(x_T)}{g(x_1)\ldots g(x_T)} = \exp(-\mu \sum_{i=1}^{T} x_i - T\mu^2/2).$$

- Clearly, $M_T > 0$ for all x and $E_{\tilde{\mathbb{P}}} M_T = 1$. The result extends to Girsanov's Theorem in continuous setting.

The Reflection Principle

Let $B_t^* = \sup_{0 \le s \le t} B_s$.

Strong Markov Property

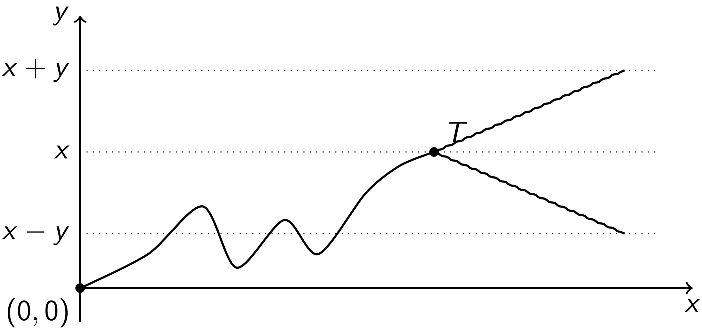
Let τ be a stopping time. The process $\hat{B}_t = B_{t+\tau} - B_\tau$ is also a Brownian motion, independent of $\mathcal{F}_\tau$.

Reflection Principle

Without loss of generality, fix $t > 0$.

$$\begin{aligned}
\mathbb{P}(B_t^* \ge x, B_t \le x - y) &= \mathbb{P}(B_t^* \ge x, B_t - B_{\tau_x} \le -y) \\
&= \mathbb{P}(B_t^* \ge x, B_t - B_{\tau_x} \ge y) \quad \text{(by symmetry)} \\
&= \mathbb{P}(B_t^* \ge x, B_t \ge x + y) \\
&= \mathbb{P}(B_t \ge x + y)
\end{aligned}$$

Reflection Principle (Visual)



The principle states that after hitting x, Path A and the reflected Path B are equally likely (in your mind, replace the lines with reflected non-differentiable Brownian paths!).

First Passage Time: CDF Derivation

From the reflection principle, $\mathbb{P}(B_t^* \geq x, B_t < x) = \mathbb{P}(B_t > x)$.

- Now, consider the probability of the maximum being at least x:

$$\mathbb{P}(B_t^* \geq x) = \mathbb{P}(B_t^* \geq x, B_t < x) + \mathbb{P}(B_t^* \geq x, B_t \geq x)$$

  The second term is just $\mathbb{P}(B_t \geq x)$ since $B_t \geq x \implies B_t^* \geq x$.

$$= \mathbb{P}(B_t \geq x) + \mathbb{P}(B_t \geq x) = 2\mathbb{P}(B_t \geq x)$$

- Let $\tau_a = \inf\{t : B_t = a\}$, for $a > 0$. The event $\{\tau_a \leq t\}$ is the same as $\{B_t^* \geq a\}$.

$$\mathbb{P}(\tau_a \leq t) = \mathbb{P}(B_t^* \geq a) = 2\mathbb{P}(B_t \geq a)$$

- The CDF is:

$$\mathbb{P}(\tau_a \le t) = 2\mathbb{P}(B_t \ge a) = 2\int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t} dz$$

- Let Φ be $N(0,1)$ CDF (ϕ denotes its pdf). Then

$$\mathbb{P}(\tau_a \le t) = 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right).$$

- The PDF $f_{\tau_a}(t)$ is the derivative with respect to t:

$$\begin{aligned} f_{\tau_a}(t) = \frac{d}{dt}\mathbb{P}(\tau_a \le t) &= -2\phi\left(\frac{a}{\sqrt{t}}\right) \cdot \left(-\frac{1}{2} a t^{-3/2}\right) \\ &= \frac{a}{t^{3/2}}\phi\left(\frac{a}{\sqrt{t}}\right) \end{aligned}$$

$$f_a(t) = \frac{a}{\sqrt{2\pi} t^{3/2}} e^{-a^2/2t}$$

First Passage Time: Expectation

Expectation

We can show that the process is certain to hit a:

$$\mathbb{P}(\tau_a < \infty) = \int_0^\infty f_{\tau_a}(t)dt = 1$$

However, the expected time to do so is infinite:

$$\mathbb{E}[\tau_a] = \int_0^\infty t \cdot f_{\tau_a}(t)dt = \int_0^\infty \frac{a}{\sqrt{2\pi t}} e^{-a^2/2t} dt = \infty$$

Alternative Definition: Gaussian Process

Gaussian Process

A process $\{X_t : 0 \leq t \leq T\}$ is a Gaussian process if for any $t_1, \ldots, t_k$, the vector $(X_{t_1}, \ldots, X_{t_k})$ has a Multivariate Gaussian (MVG(μ, Σ)) distribution.

Brownian motion as a Gaussian Process

A Gaussian process $\{X_t\}$ is a BM if:

i) $X_0 = 0$
ii) $X_t(\omega)$ is continuous in t for all $\omega \in \Omega$.
iii) $\mathbb{E}[X_t] = 0$ for all t.
iv) $\mathrm{Cov}(X_t, X_s) = \min(s, t)$.

Gaussian Process: Proof of Independent Increments

- Independent increments is the only property that needs proof.
- Let $0 \leq t_1 < t_2 < t_3 < t_4 \leq T$.

  $\mathrm{Cov}(X_{t_4} - X_{t_3}, X_{t_2} - X_{t_1})$

  $= \mathrm{Cov}(X_{t_4}, X_{t_2} - X_{t_1}) - \mathrm{Cov}(X_{t_3}, X_{t_2} - X_{t_1})$

  $= \mathrm{Cov}(X_{t_4}, X_{t_2}) - \mathrm{Cov}(X_{t_4}, X_{t_1}) - \mathrm{Cov}(X_{t_3}, X_{t_2}) + \mathrm{Cov}(X_{t_3}, X_{t_1})$

  $= t_2 - t_1 - t_2 + t_1 = 0$

- Since the increments are jointly Gaussian and have zero covariance, they are independent. Thus, $\{X_t : t \geq 0\}$ is a BM.